Inequalities and 'Maximum-Minimum' Problems

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There are many olympiad level problems in mathematics which belong to areas that are not covered well at all at schools. Three major examples are geometry, number theory, and functional equations. Such areas must be learned outside class time if one wants to be successful at solving olympiad style problems.

This set of notes considers another one of these areas: inequalities. This area is not covered well at schools mainly because there is a large variety of methods involved. One subarea of inequalities involves problems of the following form:

> "Maximise or minimise some expression, subject to some constraint."

We shall discuss some basic facts about inequalities, and then discuss these so-called 'max-min' problems.

1. Well-known inequalities

1.1 The basics

We begin with some obvious rules.

Theorem 1 *For real numbers a, b, c, d, we have the following.*

(a) *If $a > b$, then $a + c > b + c$; if $a \geq b$, then $a + c \geq b + c$.*

(b) *If $a > b$ and $c > d$, then $a + c > b + d$; if $a \geq b$ and $c > d$, then $a + c > b + d$; if $a \geq b$ and $c \geq d$, then $a + c \geq b + d$.*

(c) *If $a > b$, then $ac > bc$ if $c > 0$, and $ac < bc$ if $c < 0$; if $a \geq b$, then $ac \geq bc$ if $c > 0$, and $ac \leq bc$ if $c < 0$.*

(d) *Let $a, b, c, d > 0$. If $a > b$ and $c > d$, then $ac > bd$; if $a \geq b$ and $c > d$, then $ac > bd$; if $a \geq b$ and $c \geq d$, then $ac \geq bd$;*

(e) *$a^2 \geq 0$ always holds, with equality if and only if $a = 0$.*

(f) *Let $a, b > 0$. If $a^2 > b^2$, then $a > b$; if $a^2 \geq b^2$, then $a \geq b$.*

1.2 Proving inequalities: some well-known techniques

How do we prove an inequality at olympiad level? We will discuss many different techniques. But firstly, note that such a problem usually has two parts:

(a) Prove the inequality.

(b) If possible, state when we have equality.

It is a common mistake to ignore part (b).

1.2.1 Squares are non-negative

A complicated looking inequality can be proved if we can show that it is equivalent to an inequality of the form 'LHS ≥ 0', where 'LHS' is a sum of squares.

Example 1. Prove that $x^4 - 7x^2 + 4x + 20 \geq 0$ for every real x. When do we have equality?

We have $x^4-7x^2+4x+20 = (x^4-8x^2+16)+(x^2+4x+4) = (x^2-4)^2+(x+2)^2 \geq 0$.

For equality, we need both $x^2 - 4 = 0$ and $x + 2 = 0$. This is only possible when $x = -2$.

Example 2. Let $a, b \geq 0$. Prove that $\frac{a+b}{2} \geq \sqrt{ab}$.

Since $a, b \geq 0$, $\sqrt{a}$ and $\sqrt{b}$ are well-defined. We have $0 \leq (\sqrt{a} - \sqrt{b})^2 = a + b - 2\sqrt{ab}$, so $a + b \geq 2\sqrt{ab}$, so $\frac{a+b}{2} \geq \sqrt{ab}$. Equality holds if and only if $\sqrt{a} - \sqrt{b} = 0$, ie: when $a = b$ since $a, b \geq 0$.

Example 3. Prove that $x^2 + y^2 + z^2 \geq xy + yz + zx$ for all real x, y, z.

We have $0 \leq (x-y)^2 + (y-z)^2 + (z-x)^2 = 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx$. So $2x^2 + 2y^2 + 2z^2 \geq 2xy + 2yz + 2zx$, and we have the result by dividing by 2. Equality holds if and only if $x - y = y - z = z - x = 0$, ie: when $x = y = z$.

1.2.2 The QM-AM-GM-HM inequalities

We now describe a chain of inequalities which will be extremely useful.

Theorem 2 (The QM-AM-GM-HM inequalities) *Let $x_1, \ldots, x_n \geq 0$. We define their* quadratic mean *to be*

$$\mathrm{QM} = \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}.$$

Define their arithmetic mean *to be*

$$\mathrm{AM} = \frac{x_1 + \cdots + x_n}{n}.$$

Define their geometric mean *to be*

$$\mathrm{GM} = \sqrt[n]{x_1 \cdots x_n}.$$

For $x_1, \ldots, x_n > 0$, define their harmonic mean *to be*

$$\mathrm{HM} = \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}.$$

Then, if $x_1, \ldots, x_n > 0$, we have

$$\mathrm{QM} \geq \mathrm{AM} \geq \mathrm{GM} \geq \mathrm{HM}.$$

*If we remove '*HM*' from above, we may have $x_1, \ldots, x_n \geq 0$ instead. ie: For $x_1, \ldots, x_n \geq 0$, we have* QM $\geq$ AM $\geq$ GM.

We have QM = AM = GM = HM *if and only if $x_1 = \cdots = x_n$.*

Taking any two of these means gives us a useful inequality. For example, the 'AM-GM inequality' says: For $x_1, \ldots, x_n \geq 0$, we have

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n},$$

with equality if and only if $x_1 = \cdots = x_n$. Note that Example 2 above is the AM-GM inequality for two terms.

The 'AM-HM inequality' says: For $x_1, \ldots, x_n > 0$, we have

$$\frac{x_1 + \cdots + x_n}{n} \geq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}},$$

with equality if and only if $x_1 = \cdots = x_n$.

And so on.

Example 4. Let $a, b, c, x, y, z > 0$. Prove that

$$\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) \geq 9.$$

Applying the AM-HM inequality, we have

$$\frac{1}{3}\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right) \geq \frac{3}{\frac{x}{a} + \frac{y}{b} + \frac{z}{c}}.$$

This easily rearranges to the required inequality.

Equality holds if and only if a, b, c, x, y, z satisfy $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$.

1.2.3 Cauchy-Schwarz inequality

This is another well-known inequality which is extremely useful.

Theorem 3 (Cauchy-Schwarz inequality) *Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be real numbers. Then*

$$(x_1 y_1 + \cdots + x_n y_n)^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2).$$

Equality holds if and only if either there exists a real number k such that $x_1 = ky_1$, $x_2 = ky_2, \ldots, x_n = ky_n$, or there exists a real number ℓ such that $y_1 = \ell x_1$, $y_2 = \ell x_2, \ldots, y_n = \ell x_n$.

Note: Be careful! The statements 'there exists a real number k such that $x_1 = ky_1$, $x_2 = ky_2, \ldots, x_n = ky_n$' and 'there exists a real number ℓ such that $y_1 = \ell x_1$, $y_2 = \ell x_2, \ldots, y_n = \ell x_n$' are not quite equivalent. We will have a problem if either all $x_i = 0$ or all $y_i = 0$. For example, if $x_1 = \cdots = x_n = 0$ and $y_1, \ldots, y_n$ are arbitrary, not all zero, then the first statement above is true with $k = 0$, but the second is false!

Remarkably, almost all literature about the Cauchy-Schwarz inequality are not too careful about this, merely just stating one of these statements as the condition for equality!

Example 5. Let $x, y, z > 0$. Prove that

$$(x^2 + y^2 + z^2)^{1/2} \geq \frac{1}{13}(3x + 4y + 12z).$$

Applying Cauchy-Schwarz, we have

$$\begin{aligned}(3^2 + 4^2 + 12^2)(x^2 + y^2 + z^2) &\geq (3x + 4y + 12z)^2, \\ x^2 + y^2 + z^2 &\geq \frac{1}{169}(3x + 4y + 12z)^2.\end{aligned}$$

Since both sides are positive, the result follows by taking square roots.

Equality holds if and only if $x = 3k$, $y = 4k$ and $z = 12k$ for some real number $k > 0$.

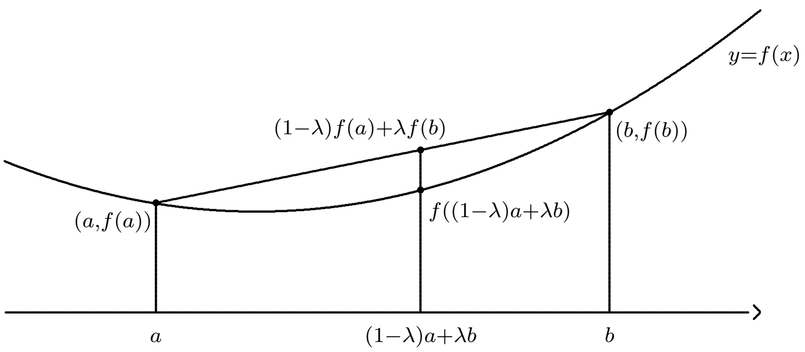
1.2.4 Jensen's inequality

This is yet another very useful inequality. To state this inequality, we must first define a property of functions.

A function $f(x)$ defined on an interval I is said to be *strictly convex* on I if for every $a, b \in I$ with $a < b$, and for every λ with $0 < \lambda < 1$, we have

$$f((1 - \lambda)a + \lambda b) < (1 - \lambda)f(a) + \lambda f(b).$$

Graphically, this just says that, no matter where we choose a and b within I, the chord joining $(a, f(a))$ and $(b, f(b))$ lies strictly above the graph of the function in the interval (a, b).



For example, $f(x) = x^2$ is strictly convex on $(-\infty, \infty)$. $g(x) = \sin x$ is strictly convex on $[\pi, 2\pi]$.

We say that $f(x)$, defined on interval I, is *strictly concave* on I if $-f(x)$ is strictly convex on I. Graphically, for all $a, b \in I$ with $a < b$, the chord joining $(a, f(a))$ and $(b, f(b))$ lies strictly below the graph of the function in the interval (a, b).

We can now state Jensen's inequality.

Theorem 4 (Jensen's inequality) *Let $f(x)$ be a function defined on an interval I, and is strictly convex on I. Then, for every $x_1, \ldots, x_n \in I$, we have*

$$\frac{1}{n}(f(x_1) + \cdots + f(x_n)) \geq f\left(\frac{1}{n}(x_1 + \cdots + x_n)\right).$$

Equality holds if and only if $x_1 = \cdots = x_n$.

A similar statement holds when we replace 'convex' by 'concave', and reverse the inequality.

Example 6. For $w, x, y, z > 0$, prove that $16(w^3+x^3+y^3+z^3) \geq (w+x+y+z)^3$. Since $f(x) = x^3$ is strictly convex on $(0, \infty)$, by Jensen's inequality, we have

$$\frac{1}{4}(w^3 + x^3 + y^3 + z^3) \geq \left(\frac{w+x+y+z}{4}\right)^3.$$

This rearranges to the required inequality. Equality holds if and only if $w = x = y = z$.

Unfortunately, it is not always easy to decide whether a function is strictly convex/concave on a certain interval. For example, where is the function $f(x) = x^4 + 4x^3 - 18x^2 + 5x$ strictly convex/concave? One way to determine this would be to use calculus, which we will come to next.

2. Calculus

2.1 Calculus or no calculus?

Despite the fact that a lot of references suggest that the knowledge of calculus is never required in olympiad style problems, calculus is still a useful tool to have. Of course, it has advantages and disadvantages. A big advantage is that it is quite powerful and sometimes it offers a 'cheap way out'; namely there may be elegant solutions to a certain problem, but calculus offers an easy, ugly solution (but still a solution!). Another is that calculus offers a way to determine where a function is strictly convex/concave. A big disadvantage is that calculus only locates local minima/local maxima/points of inflexion. A detailed analysis of the function concerned is needed to show whether a local minimum/local maximum is also indeed global. Moreover, we may need to compute the second derivative to determine the nature of stationary values, and this is not always pleasant.

Nevertheless, if the function concerned is simple enough and a detailed analysis is carried out, then calculus can offer solutions.

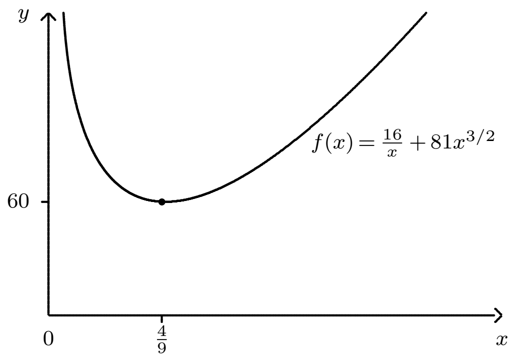
There is the following important principle in calculus as well.

Theorem 5 *Suppose $f : I \to \mathbb{R}$ is twice differentiable on the interval I. Suppose that $c \in I$ is the* only *critical point of $f(x)$. Then, $f(c)$ is the global minimum of $f(x)$ if $f''(c) > 0$, and $f(c)$ is the global maximum of $f(x)$ if $f''(c) < 0$.*

Example 7. For $x > 0$, minimise $\frac{16}{x} + 81x^{3/2}$.

At first sight, this problem is crying out for a calculus solution. Let $f(x) = 16x^{-1} + 81x^{3/2}$. We have $f'(x) = -16x^{-2} + \frac{243}{2}x^{1/2}$. Solving $f'(x) = 0$ gives $x = \frac{4}{9}$. We find that $f(\frac{4}{9}) = 60$. Also, $f''(x) = 32x^{-3} + \frac{243}{4}x^{-1/2}$. We see that $f''(x) > 0$ for all $x > 0$, so in particular, $f''(\frac{4}{9}) > 0$. Hence by Theorem 5, $f(\frac{4}{9}) = 60$ is the global minimum of $f(x)$, and 60 is the required minimum. We have equality if and only if $x = \frac{4}{9}$.

We may also argue without Theorem 5, by considering the graph of $f(x)$. Observe that $f(x) \to \infty$ when $x \to 0$ and when $x \to \infty$. Indeed $f(x)$ has the y-axis as an asymptote, and it behaves like $81x^{3/2}$ for large x. The graph of $f(x)$ looks like:



We could, in fact, use the AM-GM inequality to solve this problem: apply AM-GM on $\frac{16}{3x}$, $\frac{16}{3x}$, $\frac{16}{3x}$, $\frac{81}{2}x^{3/2}$, $\frac{81}{2}x^{3/2}$. This does not seem as obvious as the calculus approach.

As mentioned, another thing that calculus can do is to determine where a function is strictly convex/concave. We have the following result.

Theorem 6 *Suppose $f : I \to \mathbb{R}$ is twice differentiable on the interval I. Then, $f(x)$ is strictly convex on I if $f''(x) > 0$ for all $x \in I$, and $f(x)$ is strictly concave on I if $f''(x) < 0$ for all $x \in I$.*

Example 8. Determine where the function $f(x) = x^4 + 4x^3 - 18x^2 + 5x$ is strictly convex and strictly concave.

We find easily that $f''(x) = 12x^2 + 24x - 36 = 12(x + 3)(x - 1)$. We have $f''(x) > 0$ when $x < -3$ or when $x > 1$, and $f''(x) < 0$ when $-3 < x < 1$. So by Theorem 6, $f(x) = x^4 + 4x^3 - 18x^2 + 5x$ is strictly convex on $(-\infty, -3)$ and on $(1, \infty)$, and strictly concave on $(-3, 1)$.

In a nutshell, use calculus only when the function is sufficiently simple, and avoid calculus if the function is too complicated.

3. 'Maximum-Minimum Problems'

3.1 How do we tackle them?

Problems of the type

"Maximise and/or minimise (expression A), subject to (constraint B)"

can be difficult to solve. The difficulty usually seems to be to decide how to incorporate constraint B into the problem of maximising and/or minimising expression A. This gets even more difficult if, for example, expression A involves more than two variables and constraint B is only a single equation relating the variables: there may be little or no hope if we try to 'eliminate one variable', and one would have to think of another approach.

The techniques that we have already discussed more or less cover many of the strategies on how to solve such problems. One just needs to be a little clever in deciding what technique(s) to use.

Finally, once the maximum and/or minimum has/have been decided, one must then show that it/they may be attained by giving an example.

Example 9. Minimise the expression $(x+y)(y+z)$, where x, y and z are positive real numbers satisfying $xyz(x+y+z) = 1$.

We have

$$(x+y)(y+z) = xy + xz + y^2 + yz = xz + y(x+y+z) = xz + \frac{1}{xz},$$

where we have used the condition $xyz(x+y+z) = 1$ to get the last equality. Applying the AM-GM inequality gives

$$(x+y)(y+z) = xz + \frac{1}{xz} \geq 2\sqrt{xz \cdot \frac{1}{xz}} = 2.$$

To claim that '2' is indeed the required minimum value, we must give an example of x, y, z where this minimum can be attained, and satisfying the constraint. In the inequality above, we have equality when $xz = \frac{1}{xz}$, so $xz = 1$. To make things simple, choose $x = z = 1$. Then y must satisfy $y(2+y) = 1$; solving this gives $y = -1 \pm \sqrt{2}$. Since $y > 0$, we take the root with the positive sign. It is then easy to check that $(x+y)(y+z) = 2$ when $x = z = 1$ and $y = -1 + \sqrt{2}$.

Example 10. Maximise the expression $7x + 2y + 8z$, where x, y and z are positive real numbers satisfying $4x^2 + y^2 + 16z^2 = 1$.

We try to cleverly apply the Cauchy-Schwarz inequality. We have

$$(7x + 2y + 8z)^2 \leq ((2x)^2 + y^2 + (4z)^2)\left(\left(\frac{7}{2}\right)^2 + 2^2 + 2^2\right) = \frac{81}{4},$$

where we have used the condition $4x^2 + y^2 + 16z^2 = 1$ to get the last equality. We see that $7x + 2y + 8z \leq \frac{9}{2}$.

Again we must show that '$\frac{9}{2}$' can be attained by some values of x, y, z satisfying the given constraint. For equality to hold in Cauchy-Schwarz, we have $2x = \frac{7}{2}k$, $y = 2k$, $4z = 2k$ for some $k > 0$. Substituting these into $4x^2 + y^2 + 16z^2 = 1$ gives $\frac{49}{4}k^2 + 4k^2 + 4k^2 = 1$, giving $k = \frac{2}{9}$. This leads to $x = \frac{7}{18}$, $y = \frac{4}{9}$ and $z = \frac{1}{9}$. It is

then easy to check that with these values of x, y, z, we do have $4x^2 + y^2 + 16z^2 = 1$ and $7x + 2y + 8z = \frac{9}{2}$.

4. Problems

Here are some inequalities and 'maximum-minimum' problems.

1. Prove that, if x is a real number and $x \neq 0$, then
$$x^8 - x^5 - \frac{1}{x} + \frac{1}{x^4} \geq 0.$$

2. Let x and y be real numbers. Prove that $(x^3 + y^3)(x^5 + y^5) \leq 2(x^8 + y^8)$.

3. Minimise $2x^2 + y^2 + z^2$, where x, y and z are real numbers satisfying $x + y + z = 10$.

4. Let x and y be non-zero real numbers satisfying $x^2 + y^2 = 4$. Find the minimum value of
$$x^4 + \frac{1}{x^4} + y^4 + \frac{1}{y^4}.$$

5. (a) Maximise the expression $x^2y - y^2x$ when $0 \leq x \leq 1$ and $0 \leq y \leq 1$.
(b) Maximise the expression $x^2y + y^2z + z^2x - y^2x - z^2y - x^2z$ when $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $0 \leq z \leq 1$.

6. Determine the smallest value of $x^2 + 5y^2 + 8z^2$, where x, y and z are real numbers satisfying $yz + zx + xy = -1$. Does $x^2 + 5y^2 + 8z^2$ have a greatest value subject to the same constraint?

7. Let x, y and z be positive real numbers satisfying
$$\frac{1}{3} \leq xy + yz + zx \leq 3.$$
Determine the range of values of (a) xyz and (b) $x + y + z$.

8. Let x, y and z be positive real numbers satisfying $xyz = 32$. Find the minimum value of $x^2 + 4xy + 4y^2 + 2z^2$.

9. Let x and y be non-negative real numbers satisfying $x + y = 2$. Show that $x^2y^2(x^2 + y^2) \leq 2$.

10. Let x, y and z be real numbers. Prove that
$$x^6 + y^6 + z^6 + 3x^2y^2z^2 \geq 2(x^3y^3 + y^3z^3 + z^3x^3).$$

11. Let x, y and z be positive real numbers satisfying $x + y + z = 1$. Find the minimum value of
$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right).$$

12. Let x, y and z be non-negative real numbers satisfying $x + y + z = 1$. Prove that
$$7(xy + yz + zx) \le 2 + 9xyz.$$

13. Let x and y be real numbers such that $7x^2 + 3xy + 3y^2 = 1$. Show that the least positive value of $\frac{x^2+y^2}{y}$ is $\frac{1}{2}$.

14. Let x, y and z be non-negative real numbers satisfying $x + y + z = 1$. Prove that
$$x^2y + y^2z + z^2x \le \frac{4}{27}.$$

15. Let a, b, c be positive real numbers satisfying $a + b + c = 1$. Prove that
$$a^2 + b^2 + c^2 + 2\sqrt{3abc} \le 1.$$

16. Let a, b, c be real numbers such that $a, b, c \ge -\frac{3}{4}$ and $a+b+c = 1$. Prove that
$$\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \le \frac{9}{10}.$$

17. Let a, b, c, d be positive real numbers satisfying $ab + bc + cd + da = 1$. Prove that
$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \ge \frac{1}{3}.$$

18. Let a, b, c be positive real numbers satisfying $abc = 1$. Prove that
$$\frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca} \le 1.$$